problem, we obtained the following upper and lower bounds on the volume of 8 :

$$9{:}070e{-}2 \le \text{Vol}{.}\,8\,/ \le 9{:}152e{-}2 \tag{8}$$

and the following bounds on the volume of $8_{\,Q}$:

$$5{:}254e{-}2 \le \text{Vol}{.}\,8_{\,Q'} \le 5{:}466e{-}2 \tag{9}$$

Therefore the ratio of the volume of $\mathcal{E}$ to $8_{\,Q}$ was found to be bounded as

$$68{:}52\% \ge \frac{\text{Vol}{.}\,\mathcal{E}_{P'}}{\text{Vol}{.}\,8_{\,Q'}} \ge 65{:}86\% \tag{10}$$

and the ratio of the volumes of $8_{\,Q}$ to 8 was found to be bounded as

$$60{:}3\% \ge \frac{\text{Vol}{.}\,8_{\,Q'}}{\text{Vol}{.}\,8\,/} \ge 57{:}40\% \tag{11}$$

This ratio may be compared with a comparable volume ratio of 67:1% using the methods proposed by Bordignon and Durham[13] (an 11% difference in volume). Since the volumes represent distances cubed, the 57:43% and 67:1% volume ratios approximately represent distance ratios of 83:12% and 87:55%. This is a difference of 5:3%. Although the presented method does not achieve the same level of performance as earlier methods[13] (at least in the given numerical example), it still has desirable features: Because it is the solution to a semidefinite, convex optimization program, its resolution time is very predictable ahead of time, which might be of interest to on-line reconfiguration, or as a first guess for existing methods.[13]

IV. Conclusions

In this Note, we have considered the control surface allocation problem in the case when the surface allocation is limited to be a linear mapping from moment space to control space. We have shown that an approach to that problem based on ellipsoid volume maximization can be easily recast as a convex optimization problem. This method has been applied to a numerical model of the F-18 HARV and has been compared with other approaches. The convex nature of the optimization problem under consideration makes it possible to incorporate the proposed procedure in a real-time aircraft control allocation reconfiguration in the event of damaged control surfaces. The by-products of the optimization procedure (especially the resulting ellipsoids) may be used in other proposed surface allocation procedures as well.

References

[1]Shewchun, J. M., and Feron, E., "High Performance Bounded Control for Systems Subject to Actuator Amplitude and Rate Saturation," *International Journal on Robust and Nonlinear Control* (to be published).

[2]Saberi, A., Lin, Z., and Teel, A. R., "Control of Linear Systems with Saturating Actuators," *IEEE Transactions on Automatic Control*, Vol. 41, No. 3, 1996, pp. 368–378.

[3]Chernousko, F. L., *State Estimation for Dynamic Systems*, CRC Press, Boca Raton, FL, 1994.

[4]Gavrilyako, V. M., Korobov, V. I., and Skylar, G. M., "Designing a Bounded Control of Dynamic Systems in Entire Space with the Aid of a Controllability Function," *Automation and Remote Control*, Vol. 47, No. 11, 1987, pp. 1484–1490.

[5]Kamenetskii, V. A., "Synthesis of Bounded Stabilizing Control for an n-Fold Integrator," *Automation and Remote Control*, Vol. 52, No. 6, 1991, pp. 770–775.

[6]Komarov, V. A., "Design of Constrained Controls for Nonautonomous Linear Systems," *Automation and Remote Control*, Vol. 45, No. 10, 1984, pp. 1280–1286.

[7]Lin, Z., and Saberi, A., "Semi-Global Stabilization of Partially Linear Composite Systems via Linear High and Low Gain State Feedback," *Proceedings of the American Control Conference* (San Francisco, CA), American Automatic Control Council, Evanston, IL, 1993, pp. 1184, 1185.

[8]Lin, Z., "Semi-Global Stabilization of Linear Systems with Position and Rate-Limited Actuators," *Systems and Control Letters*, Vol. 30, No. 1, 1997, pp. 1–11.

[9]Megretski, A., "Output Feedback Stabilization with Saturated Control: Making the Input-Output Map L2 Bounded," *Proceedings of the 13th IFAC World Congress* (San Francisco, CA), International Federation on Automatic Control, Laxemburg, Austria, 1996, pp. 435–440.

[10]Durham, W. C., "Constrained Control Allocation: Three-Moment Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 2, 1994, pp. 330–336.

[11]Enns, D. F., "Control Allocation Approaches," *Proceedings of the AIAA Conference on Guidance, Navigation, and Control* (Boston, MA), AIAA, Reston, VA, 1998, pp. 98–108.

[12]McGovern, L., and Feron, E., "Requirements and Hard Computational Bounds for Real-Time Optimization Problems," *Proceedings of the IEEE Conference on Decision and Control* (San Diego, CA), Inst. of Electrical and Electronics Engineers, New York, 1998, pp. 3366–3371.

[13]Bordignon, K. A., and Durham, W. C., "Closed-Form Solutions to Constrained Control Allocation Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 5, 1995, pp. 1000–1007.

[14]Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, Vol. 15, SIAM Studies in Applied Mathematics, Society of Industrial and Applied Mathematics, Philadelphia, PA, 1994.

[15]John, F., "Extremum Problems with Inequalities as Subsidiary Conditions," *Fritz John, Collected Papers*, edited by J. Moser, Birkhauser, Boston, MA, 1985, pp. 543–560.

[16]Nesterov, Yu., and Nemirovsky, A., *Interior-Point Polynomial Methods in Convex Programming*, Vol. 13, SIAM Studies in Applied Mathematics, Society of Industrial and Applied Mathematics, Philadelphia, PA, 1994.

[17]Wu, S., and Boyd, S., "SDPSOL, A Parser/Solver for Semidefinite Programming and Determinant Maximization Problems with Matrix Structure," Stanford Univ., Stanford, CA, May 1996.

New Constraint Stabilization Technique for Dynamic Systems with Nonholonomic Constraints

S. D. Wu,* J. C. Chiou,† and J. Y. Yang*

National Chiao Tung University, Hsinchu 300, Taiwan, Republic of China

Introduction

IN the last two decades, computer simulation of multibody dynamic (MBD) systems has enjoyed substantial progress to design and analysis engineering problems such as robot arm manipulators and space vehicles. In general, the dynamic equations of MBD systems with holonomic and/or nonholonomic constraints can be derived and expressed in a set of differential algebraic equations (DAEs). Because the solution procedure of DAEs suffer drawbacks such as constraint violation and numerically stiff in the computer implementation, these have motivated researchers to look for alternative solution procedures that overcome the preceding difficulties. Conventional nonholonomic constraint stabilization techniques such as the stabilization technique by Baumgarte,[1,2] the penalty method by Orlandea et al.[3] and Lötstedt[4] and the penalty staggered stabilized procedure by Park and Chiou[5] are developed to correct the constraint forces during the process of numerical integration. However, these techniques require one to choose suitable stabilization parameters for their different applications. In general, different parameters will cause different results in correcting constraint violations. Moreover, the simulated DAE will become numerically unstable if inappropriate parameters are chosen. To obtain a parameter-free

Received Sept. 29, 1998; revision received Dec. 15, 1998; accepted for publication Dec. 18, 1998.

*Research Assistant, Department of Electrical and Control Engineering.

†Associate Professor, Department of Electrical and Control Engineering. E-mail: chiou@cc.nctu.edu.tw. Member AIAA.

numerical simulation environment, a stabilization technique that requires no stabilization parameter is proposed. The technique that is based on so-called velocity correction method is used to correct the constraint errors effectively and efficiently.

Equations of Motion of Dynamic Systems with Nonholonomic Constraints

In general, the equations of motion for dynamical systems with nonholonomic constraints can be derived and expressed in various forms depending on the method used to describe the systems. Here, the equations of motion of dynamic systems with nonholonomic constraints are expressed as[5]

$$M\ddot{q} + B^T\lambda = F \tag{1}$$

$$\Phi_{nh}(q, \dot{q}, t) = B\dot{q} + c(q, t) = 0 \tag{2}$$

where M denotes the mass matrix of the system; q are the generalized coordinate components; λ is the Lagrange multiplier; B is the Jacobian matrix of the constraint equations; F consists of the applied force, the centrifugal, Coriolis acceleration, and the internal spring force; and Φ_{nh} are the nonholonomic constraints.

Baumgarte Constraint Stabilization Technique

Because the solution procedure of Eqs. (1) and (2) suffer various drawbacks such as constraint violation in the computer implementation,[3,4] motivated researchers such as Baumgarte are looking for alternative solution procedures that can overcome these drawbacks. In the Baumgarte technique, one replaces Eq. (2) by

$$\dot{\Phi}_{nh} + \gamma\Phi_{nh} = 0 \tag{3}$$

where γ are real numbers that needed to be determined. Substituting Eq. (3) into Eq. (2) and replacing $\ddot{q}$ from the equations of motion, the Lagrange multiplier for nonholonomic is obtained. As the numerical integration proceeds, the constraints may be violated because of the local truncation error of the employed integrators. In such a case, the constraint violation is amplified by the chosen stabilization parameter γ and feedback to Eq. (3). Note that for different stabilization parameters, different constraint errors occurred in simulating MBD systems. Furthermore, the simulated dynamic system may become numerically unstable if an inappropriate parameter γ is chosen.

Velocity Correction Method

The Baumgarte technique stabilizes the constraint error by correcting the Lagrange multipliers (or constraint forces). They encounter the same drawback: the constraint error does not equal zero exactly. To overcome this drawback, a new stabilization procedure called the velocity correction method is proposed. The basic concept of the present method is motivated by the principle of energy conservation that is given as follows:

$$H(q, \dot{q}) = \tfrac{1}{2}\dot{q}^T M\dot{q} + U(q) = H_0 \tag{4}$$

where $H(q, \dot{q})$ define a system energy function and H_0 is the initial energy. One can use the following equation to update velocity and preserve the system energy exactly:

$$\dot{\tilde{q}} = \sqrt{\frac{H_0 - U(q)}{\frac{1}{2}\dot{q}^T M\dot{q}}}\,\dot{q} \tag{5}$$

Combining Eqs. (4) and (5), one obtains

$$H(q, \dot{q}) = \tfrac{1}{2}\dot{\tilde{q}}^T M\dot{\tilde{q}} + U(q) = H_0 \tag{6}$$

The same technique can also be applied to stabilize dynamic system with nonholonomic constraints. To develop the proposed technique, we assume that $C_t = 0$, which yields

$$\Phi_{nh} = B\dot{q} = 0 \tag{7}$$

Numerically speaking, Eq. (7) is not always satisfied. In fact, because of the truncation error of employed integrators, Eq. (7) always yields a constraint error, $\xi(t)$, i.e.,

$$\Phi_{nh} = B\dot{q} = \xi(t) \tag{8}$$

where $\xi(t)$ depends on the employed integrators, the integration step size, and the stabilization procedure. If Eq. (7) is used to stabilize the constraint errors, $\xi(t)$ will diverge because of the numerical truncation errors. Now, if we assume that the constraint errors are caused by the velocity truncation errors, then we conclude that the corrected velocities $\dot{\tilde{q}}$ are the summation of the calculated velocity $\dot{q}$ and the truncation error of velocity $\dot{q}_e$:

$$\dot{\tilde{q}} = \dot{q} + \dot{q}_e \tag{9}$$

The object of the present technique is to satisfy the following equation:

$$B\dot{\tilde{q}} = 0 \tag{10}$$

Substituting Eq. (9) into Eq. (8), one obtains

$$B(\dot{\tilde{q}} - \dot{q}_e) = \xi(t) \tag{11}$$

Combining Eqs. (10) and (11) yields

$$B\dot{q}_e = -\xi(t) \tag{12}$$

The minimum norm solution of Eq. (12) can be obtained as follows[6]:

$$\dot{q}_e = -B^T(BB^T)^{-1}\xi(t) \tag{13}$$

Substituting Eq. (8) into Eq. (13) and combining Eq. (9), we have

$$\dot{\tilde{q}} = \dot{q} + \dot{q}_e = [I - B^T(BB^T)^{-1}B]\dot{q} \tag{14}$$

For the general case of nonholonomic constraint,

$$\Phi_{nh} = B\dot{q} + c_t = 0 \tag{15}$$

Following the procedure from Eq. (7) to Eq. (14), the correcting velocity can be derived as

$$\dot{\tilde{q}} = \dot{q} - B^T(BB^T)^{-1}(B\dot{q} + c_t) \tag{16}$$

Remark 1: No matter what the integrators and integration step size used, the constraint error is always equal to the machine constant.

Remark 2: If the simulated dynamic system is an energy conserving system, we can combine Eq. (14) and Eq. (5) to stabilize nonholonomic constraint and energy constraint, i.e.,

$$\dot{\tilde{q}} = \dot{q} + \dot{q}_e = [I - B^T(BB^T)^{-1}B]\dot{q} \tag{17a}$$

$$\dot{\hat{q}} = \sqrt{\frac{H_0 - U(q)}{\frac{1}{2}\dot{\tilde{q}}^T M\dot{\tilde{q}}}}\,\dot{\tilde{q}} \tag{17b}$$

where $\hat{\dot{q}}$ is the final computed velocity vector that preserves the energy constraint. To justify the proposed constraint stabilization technique for a dynamic system with nonholonomic constraints, the following numerical example is given.

Numerical Example: Rolling Wheel with Steering and Driving Inputs[7]

Let x and y denote the coordinates of the point of contact of the wheel on the plane, let ϕ denote the heading angle of the wheel, measured from the x axis, and let θ denote the rotation angle of the wheel because of rolling, measured from a fixed reference. Let u_1 denote the applied torque about the rolling axis of the wheel and let u_2 denote the applied torque about the vertical axis through the point of contact.

The equations of motion of the present example is characterized by the following matrices, constraints, and input forces:

$$M = \mathrm{diag}[1, 1, 1, 1], \qquad q = [x \quad y \quad \theta \quad \phi]^T$$

$$B = \begin{bmatrix} 1 & 0 & -\cos\phi & 0 \\ 0 & 1 & -\sin\phi & 0 \end{bmatrix}, \qquad F = [0 \quad 0 \quad u_1 \quad u_2]^T$$

$$u_1 = 2\cos(t), \qquad u_2 = \sin(t)$$

The initial conditions are given as

$$x(0) = 0, \qquad y(0) = 0, \qquad \phi(0) = -1, \qquad \theta(0) = 0$$

$$\dot{x}(0) = 0, \qquad \dot{y}(0) = 0, \qquad \dot{\phi}(0) = 0, \qquad \dot{\theta}(0) = -1$$

The Adams–Bashforth third-order (AB-3) integration method is used in the simulations with step size $h = 0.01$ s. The starting procedure of AB-3 is resolved by using Runge–Kutta forth-order integration. Figures 1 and 2 illustrate the numerical performances of the Baumgarte technique and the velocity correction method. The stabilization parameters for the Baumgarte technique are chosen to be $\gamma = 5$ and $\gamma = 50$ (optimal[8]), respectively. As shown in Fig. 1, by using these parameters, the 2 norm of constraint errors are oscillating within a specific interval. (The interval is dependent on the stabilization parameter and integration step size.) In Fig. 2, we observe that the constraint error for the velocity correction method is almost equal to machine constant. In comparing Figs. 1 and 2, it should be pointed out that the constraint error of the Baumgarte technique can be improved with a different choice of γ and that the nice feature of the proposed method is merely a result of enforcing the minimum norm solution of the constraint error equation. Although not reported here, different types of integration algorithms were also conducted to examine the effectiveness of the proposed method. Similar results, as shown in Figs. 1 and 2, were obtained.

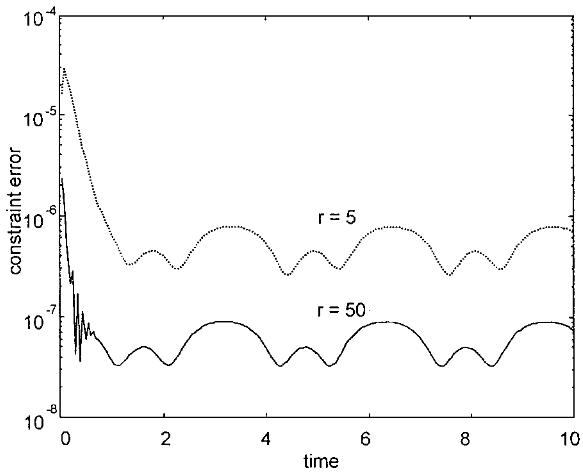

Fig. 1 Calculated constraint error using the Baumgarte technique.

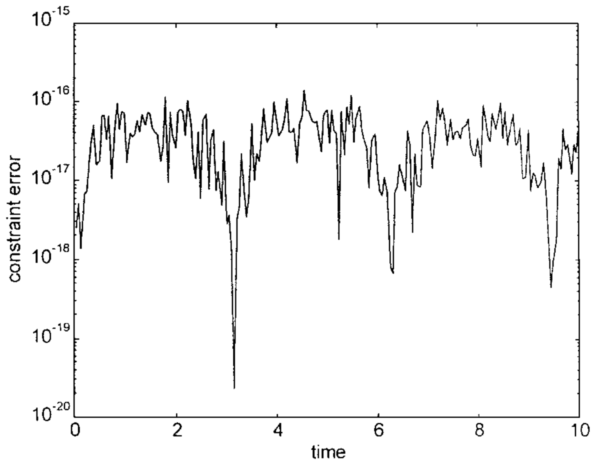

Fig. 2 Calculated constraint error using the velocity correction method.

Thus, we conclude that the velocity correction method is more efficient than the Baumgarte technique.

Conclusion

A new constraint stabilization technique, based on the velocity correction method, is proposed for solving MBD systems with nonholonomic constraints. The technique is based on the assumption that the numerical errors occurred in updating both the nonholonomic constraint equation and the numerically integrated velocity. By propagating the velocity errors into a nonholonomic constraint equation, a modified velocity vector that consists of constraint Jacobian matrix and original calculated velocity is obtained. Unlike the conventional constraint stabilization technique, the present method does not require one to choose parameters that greatly affect the performance in simulating MBD systems. A computational procedure based on this finding is developed to suppress the constraint violation during the process of numerical integration. A numerical example indicated that the constraint errors of nonholonomic constraint are closed to the machine constant for the several chosen numerical time integrators.

Acknowledgment

The present work was supported by the National Science Council under Grant NSC 86-2213-E-009-043. This paper was presented at the Nonlinear Theory and Its Applications Conference, Crans-Montana, Switzerland, Sept. 14–17, 1998.

References

[1]Baumgarte, J. W., "Stabilization of Constraints and Integrals of Motion in Dynamical System," *Computational Method in Applied Mechanics and Engineering,* Vol. 1, No. 4, 1972, pp. 1–16.

[2]Baumgarte, J. W., "A New Method of Stabilization for Holonomic Constraints," *Journal of Applied Mechanics,* Vol. 50, Dec. 1983, pp. 869, 870.

[3]Orlandea, N., Chase, M. A., and Calahan, D. A., "A Sparsity-Oriented Approach to the Dynamic Analysis and Design of Mechanical Systems—Parts I and II," *Transactions of the ASME, Journal of Engineering for Industry,* Series B, Vol. 99, Aug. 1977, pp. 773–784.

[4]Lötstedt, P., "On a Penalty Function Method for the Simulation of Mechanical Systems Subject to Constraints," Royal Institute of Technology, TRITA-NA-7919, Stockholm, Sweden, 1979.

[5]Park, K. C., and Chiou, J. C., "Stabilization of Computational Procedures for Constrained Dynamical Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 4, 1988, pp. 365–370.

[6]Junkins, J. L., and Kim, Y., *Introduction to Dynamics and Control of Flexible Structures,* AIAA Education Series, AIAA, Washington, DC, 1993, pp. 12, 13.

[7]Block, A. M., and McClamroch, N. H., "Control and Stabilization of Nonholonomic Dynamic Systems," *IEEE Transactions on Automatic Control,* Vol. 37, No. 11, 1992, pp. 1746–1757.

[8]Chiou, J. C., Yang, J. Y., and Wu, S. D., "Stability Analysis of Baumgarte Constraint Stabilization Technique in Multibody Dynamic Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 22, No. 1, 1999, pp. 160–162.